2 Theory of complexity

How difficult are the problems to solve? How

some machine models allow faster solution than the others?

Example: Given NEA A. Is it true that $L(A) = \Sigma^*$?

Terminology and Convention

n: denotes the size of the input

Unit is to be still specified. bit, tape symbols, RAM machine word.

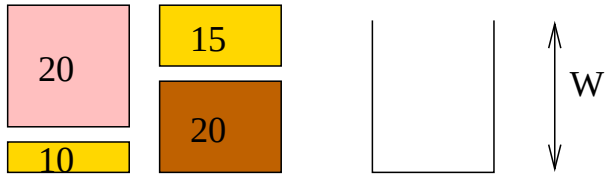
(decision) ”Problem” : One language which can be recognized.

We talk here only about decidable problems.

Complexity measures

- Time. Here is the main thing
- Space
- Energy consumption
- Communication capacity
- Disk access
- Chip surface for Hardware implementing
- ...

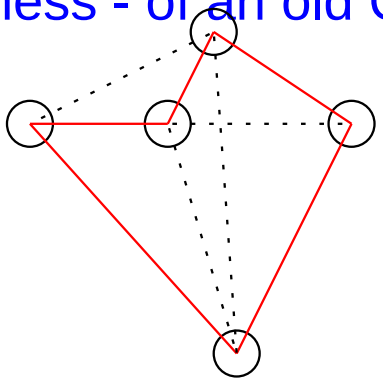
Example Knapsacks problem



- Given n things with weight $w_i \in \mathbb{N}$ and profit p_i
- Is there a subset $\mathbf{x}$ of objects
- so that $\sum_{i \in \mathbf{x}} w_i \leq W$ and
- maximize the profit $\sum_{i \in \mathbf{x}} p_i$

Example: Travelling Salesman Problem (TSP)

[The TSP- which it has to do, in order to receive orders and be certain a happy success in its business - of an old Commis Voyageur, 1832].



Given a graph $G = (V, V \times V)$, find a simple cycle $C = (v_1, v_2, \ldots, v_n, v_1)$ so that $n = |V|$ and $\sum_{(u,v)\in C} d(u,v)$ is minimal.

Formulate as decision problem: as we had

Hamilton cycle problem

[Hamilton, William Rowan, Memorandum respecting a new system of roots of unity. Philosophical Magazine, 12 1856]

$M := \{G = (V,E) : \exists C \subseteq E : |C| = |V|, C \text{ is a simple cycle}\}$

Coding a graph $G = (V,E)$ as a word of $\{0,1,\#\}^*$:

$V \subseteq \{0,1\}^{\lceil \log |V| \rceil}$

Coding w_G:= $\prod_{(u,v)\in E} u\#v\#$

Minimal Steiner tree

[C. F. Gauss 18??] Combinatorial optimization.

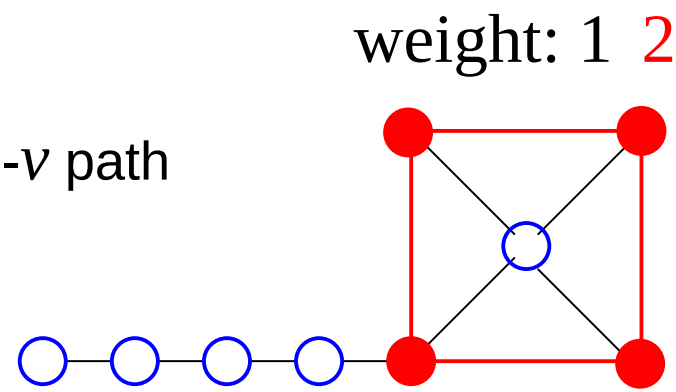
Given a graph $G = (V, E)$, with positive weighted edges $c : E \to \mathbb{R}_+$

$V = R \cup F$, i.e., Obligatory nodes and Steiner nodes

find a tree $T \subseteq E$ with a minimal cost and all obligatory nodes are connected.

$\forall u, v \in R : T$ exists u-v path

The network design problem



Upper bounds

Algorithms given and analysed.

Problem:

- Is the analysis exactly enough?
- Are there better algorithms?

Lower bounds

No algorithm could a better solution.

Trivial lower bounds :

$T = \mathcal{O}(\text{inputSize} + \text{outputSize})$

Problem: Hardly better bounds known!

We have to predicate over all algorithms!

Exceptions:

additional assumptions: for example $\Omega(n \log n)$ for comparative sorting of n elements

very limited models: for example $\Omega(n^2)$ for one tape TM-acceptor for $L_P := \{w : w = w^R\}$. Communications complexity argument collapses already by 2 tapes

Lower bounds solution

- Abstraction: Ignore smaller differences
- Classification: A set for problems that are ”almost equally difficult“.

A complexity class

$\text{time}_M(w) =$ Number of calculation steps of one TM M by an input w

$\text{TIME}(f(n)) =$

$\{L : \exists TM\ M : L(M) = L \wedge \forall w \in \Sigma_M^* : \text{time}_M(w) \leq f(|w|)\}.$

Here many-tapes TM.

Polynomial

A function $p : \mathbb{N} \to \mathbb{N}$ of the form

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$$

with a_i, $k \in \mathbb{N}$

Complexity class P

$$\mathbf{P} := \bigcup_{\text{Polynomial } p} \text{TIME}(p(n))$$

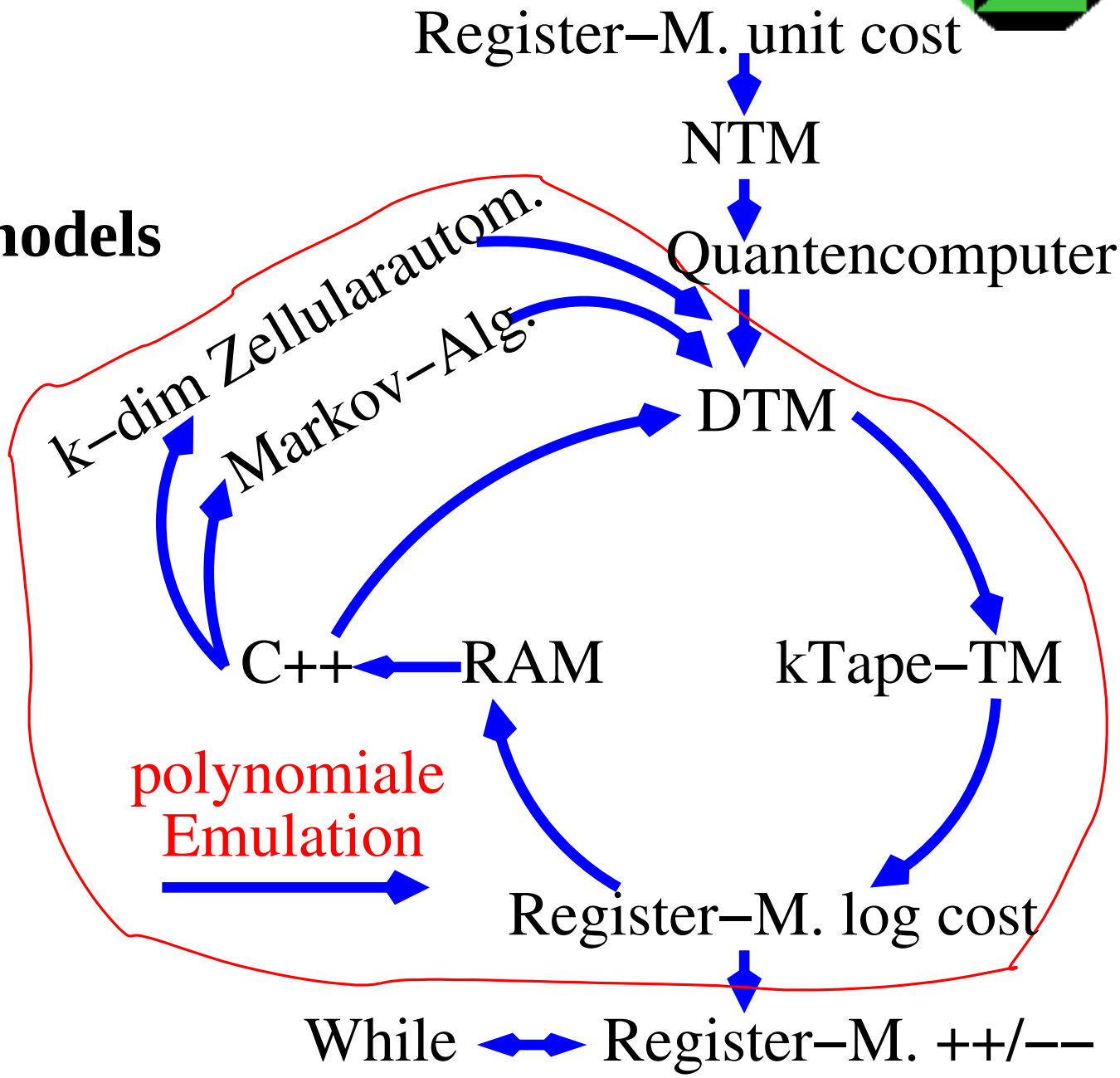
By analogy we define in polynomial time computable function.

Complexity class P

$$\mathbf{P} := \bigcup_{\text{Polynomial } p} \text{TIME}(p(n))$$

Interpretation/Convention: Problem in **P** is efficiently solvable

P for different machine models



Transformation optimizing problem $\rightarrow$ decision-making problem

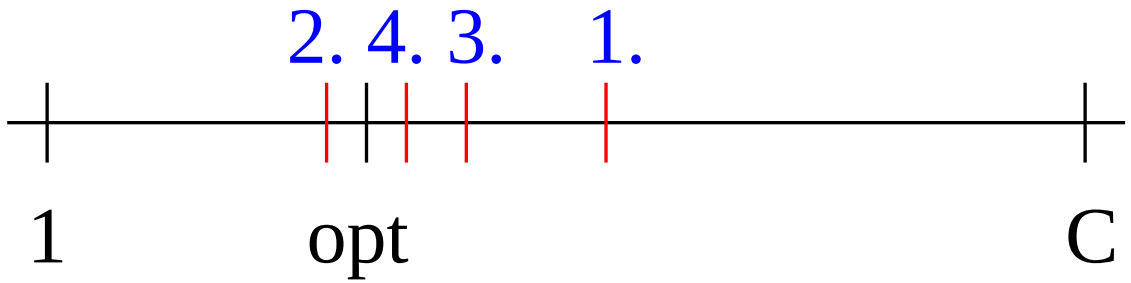
Assume: Target function value $\in 1..C$, whole-numbered.

Binary Search: $\lceil \log C \rceil$ Decision problem solved.

Polynomial in n, when $\log C$ polynomial in n.

Also if C has polynomial many bits.

This is the price of output complexity!



Another complexity class

Let M be a non-deterministic Turing machine

$$\text{ntime}_M(w) := \begin{cases} \min\{|P| : P = (s)w \vdash^* u(f)v, f \in F\} & \text{if } w \in L(M) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{NTIME}(f(n)) := \{L : \exists NTM\ M : L(M) = L \wedge \forall w \in \Sigma_M^* : \text{ntime}_M(w) \leq f(|$$

Complexity class NP

$$\mathbf{NP} := \bigcup_{\text{Polynomial } p} \text{NTIME}(p(n))$$

Example: Knapsack problem

// Is there $x_1 \cdots x_n \in \{0,1\}^n : \sum_i x_i w_i \leq W \wedge \sum_i x_i p_i \geq P$?

Procedure knapsack$(\langle w_1, \ldots, w_n \rangle, \langle p_1, \ldots, p_n \rangle, W, P)$

for $i := 1$ **to** n **do** nondeterministically guess $x_i \in \{0,1\}$

if $\sum_i x_i w_i > W$ **then** reject

if $\sum_i x_i p_i < P$ **then** reject

accept

Knapsack$\in NP$

Alternative definition for NTIME: oracle

A DTM M oracle-accepted $w \in \Sigma^*$ in time $t = \text{otime}_M(w)$ if

$\exists o \in \Gamma^*$: M started by $o(s)w$

halts in t transitions

in a configuration $x(f)y$ with $f \in F$.

If $w \in \Sigma^*$ for M is not oracle-accepted, then $\text{otime}_M(w) := 0$.

$\text{OTIME}(f(n)) := \{L : \exists DTM\ M : L(M) = L \wedge \forall w \in \Sigma_M^* : \text{otime}_M(w) \leq f(|$

Equvalence for NTIME and OTIME

NTM emulates DTM with oracle:

Nondeterministic computation „advises“ the oracle *o*

oracle-TM emulates NTM:

oracle gives the nondeterministic decisions.

The 1 000 000 $ question

$$\mathbf{P} = \mathbf{NP}?$$

One of 7 mathematical problems for which the

Clay Mathematics Institute

a price of

1 000 000 US$

is announced.

Observation: $\mathbf{P} \subseteq \mathbf{NP}$

Question for the public

100 researchers are questioned for $\mathbf{P} = \mathbf{NP}$

61: No

09: Yes

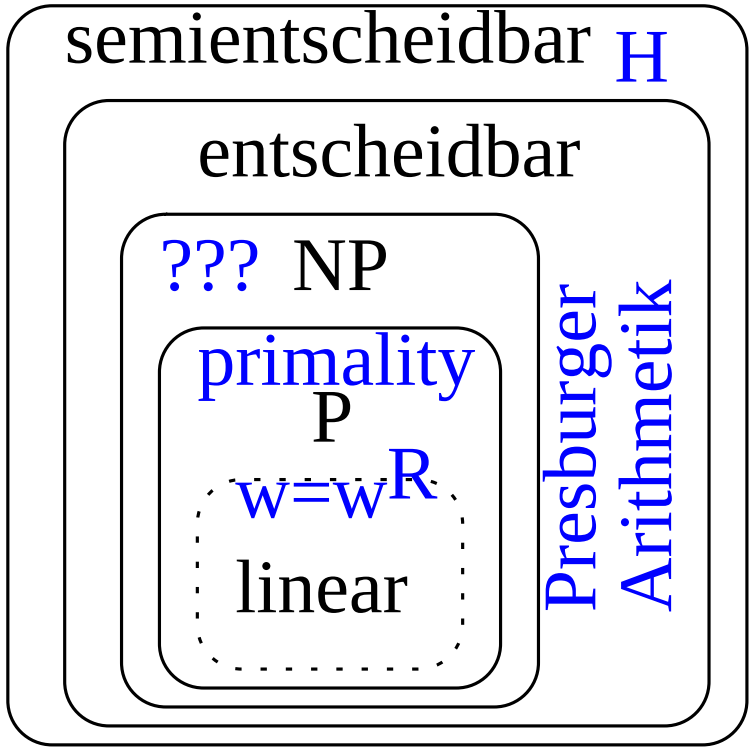
22: don't know

08: no answer

(independently of the accepted axioms.)

Way does this question so important?

A complexity hierarchy



Presburger Arithmetic

The decision problem for the first order calculus with the following restriction:

- Constants 0, 1
- Variables $x_i \in \mathbb{Z}$
- Functions $+$, $-$
- Relations $<$, $=$
- Logical connectives $\wedge$, $\vee$, $\neg$
- Quantifiers$\exists$, $\forall$

Polynomial reductions

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages.

$A \leq_p B$ (A is polynomial reducible to B)

$\Leftrightarrow$

$\exists f : \Sigma^* \to \Gamma^* : \forall w \in \Sigma^* : w \in A \Leftrightarrow f(w) \in B$

where f is in polynomial time computable.

Example

Theorem: Hamilton Cycle$\leq_p$TSP

Proof:

Let $G = (V, E)$ be a non directed graph.

Define $d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 1+\alpha & else \end{cases}$

Then, when G has a Hamilton cycle

$\exists$ TSP tour with cost n

(otherwise optimal cost $\geq n + \alpha$)

Let $G = (V, E)$ be an arbitrary undirected graph.

$$d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 1+\alpha & else \end{cases}$$

$\exists$ round tour C with weight n

$\longrightarrow$ No edges in C have weight > 1

$\longrightarrow$ All edges in C have weight 1

$\longrightarrow$ all edges in C are edges in G

$\longrightarrow$ C has a Hamilton cycle in G

Let $G = (V, E)$ be an arbitrary undirected graph.

$$d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 1+\alpha & else \end{cases}$$

G has Hamilton cycle C

$\longrightarrow C$ is a round tour with weight n.

Lemma: $A \leq_p B, B \in \mathbf{P} \rightarrow A \in \mathbf{P}$

Proof:

Let $A \leq_p B$ by function f.

Let M_f be a TM, such that f is computable by polynomial time bound p.

Further let $B \in \mathbf{P}$ by polynomial time bound q by TM M_B.

Consider compositionsTM $M_A := (M_f; M_B)$.

M_A decides A.

The time calculation for an input w:

$p(|w|) + q(|f(w)|) \leq p(|w|) + q(|w| + p(|w|))$

It is polynomial in $|w| = n$

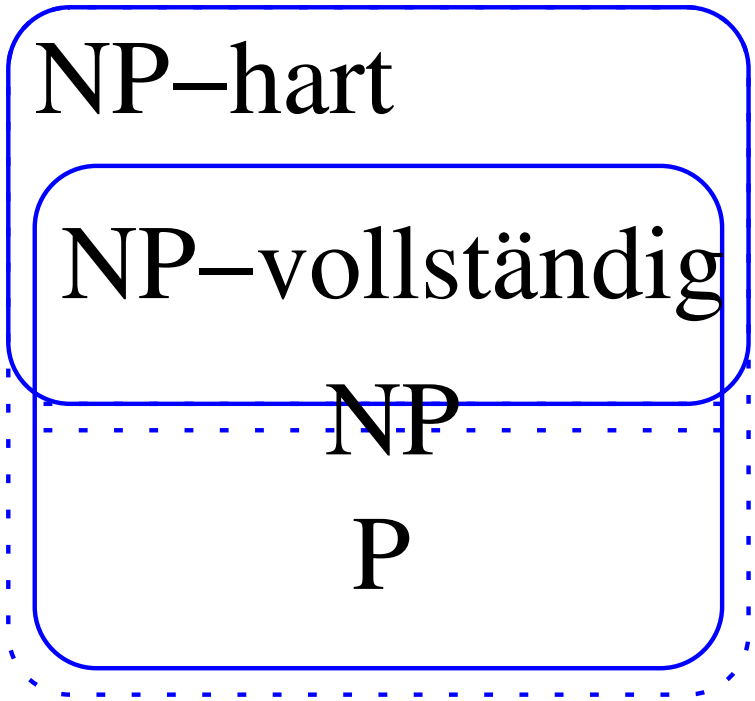
Lemma: $A \leq_p B, B \in \mathbf{NP} \rightarrow A \in \mathbf{NP}$

Proof: analog.

NP-hard and NP-complete problems

A is **NP**-hard:$\Leftrightarrow \forall L \in \mathbf{NP} : L \leq_p A$

A is **NP**-complete:$\Leftrightarrow A$ is **NP**-hard and $A \in \mathbf{NP}$.



A simple way to Glory and Richness?

Proposition: Let A be **NP**-complete. Then: $A \in \mathbf{P} \Leftrightarrow \mathbf{P} = \mathbf{NP}$

Proof:

Case $\mathbf{P} = \mathbf{NP}$:

$A \in \mathbf{NP} = \mathbf{P}$ also in particular $A \in \mathbf{P}$

Case $A \in \mathbf{P}$:

Let $L \in \mathbf{NP}$ be an arbitrary.

Since A is **NP**-hard then $L \leq_p A$

and because of $A \in P$ it follows $L \in \mathbf{P}$

Also $\mathbf{P} = \mathbf{NP}$

SAT: The satisfiability problem

given: Formula F in the Boolean logic

($\wedge \vee \neg \rightarrow ()$, Variables)

question: is F satisfiable?, i.e.

$\exists$ evaluation of the variables in $\{0,1\}$ so that the truth value of $(F) = 1$?

Formally:

SAT

$:= \{\text{code}(F) \in \Sigma^* : F$ is satisfiable formula in the Boolean logic$\}$,

code is an appropriate coding of the formulas as strings.

Theorem: [Cook 1971] and [Levin 1971] SAT is **NP**-complete.

Proof for SAT$\in$ NP

Procedure satisfiable(F)

guess values $\in \{0,1\}$ for all variables in F

substitute variables by their values

evaluate the resulting truth value v of F

if $v = 1$ **then** accept

Proof of that SAT is NP-hard.

Let

$L \in$ **NP** arbitrary,

$M = (\{1,\ldots,k\},\Sigma,\{1,\ldots,\ell\},\delta,1,H)$ NTM where $L(M) = L$,

($\sqcup = \ell$)

$p(n)$ polynomial time bound for acceptance of L by M for,

$w = w_1 w_2 \cdots w_n \in \Sigma^*$ arbitrary input.

Basic approach: We show that $L \leq_p$SAT,

by constructing a formula F of polynomial length, so that

$$w \in L \Leftrightarrow F \text{ is satisfiable}$$

Variables for F

- $\text{state}_{tz} = 1 \Leftrightarrow$ after t steps M is in state z
- $\text{pos}_{ti} = 1 \Leftrightarrow$ after t steps M is in a tape position i
- $\text{band}_{tia} = 1 \Leftrightarrow$ after t steps the tape position i is labelled by a

Notice that

$$0 \leq t \leq p(n) \text{ and}$$
$$-p(n) \leq i \leq p(n)+1$$

There are only $\mathcal{O}\left(p(n)^2\right)$ variables

The architecture of $F =$

$R\wedge$ „The var. describe always one configuration“

$A\wedge$ „The var. describe the initially configuration $(1)w$“

$U_1\wedge$ „At the head position the correspond modifications as δ says“

$U_2\wedge$ „At non head positions nothing is changing“

E „M accepts w in $\leq p(n)$ steps“

Proof $w \in L \longrightarrow F$ is satisfiable

$w \in L$

$\longrightarrow \exists$ a sequence of configurations $P = (1)w \vdash^* u(h)v$ with $h \in H$ for M, $|P| = p(n)$

P defines a variable assignment so that F to get the truth value 1.

$\longrightarrow F$ is satisfiable.

Proof F satisfiable $\longrightarrow w \in L$

F satisfiable by assigned truth values of variables $\longrightarrow$

- R will be fulfilled $\longrightarrow$

  The Var. describe at each time a configuration

- A will be fulfilled $\longrightarrow$

  The var. describe initially the configuration $(1)w$

- U_1, U_2 will be fulfilled $\longrightarrow$

  from t to $t+1$ there is a transition according δ

- E will be fulfilled $\longrightarrow$

  the computation ends in one accepting state.

$\longrightarrow w \in L$

There can be only one

$G(v_1, \ldots, v_m) = 1 \Leftrightarrow$ Exactly one $v_i = 1$.

Implementation:

$G(v_1, \ldots, v_m) =$

$v_1 \wedge \neg v_2 \wedge \neg v_3 \wedge \cdots \wedge \neg v_m \vee$

$\neg v_1 \wedge v_2 \wedge \neg v_3 \wedge \cdots \wedge \neg v_m \vee$

$\neg v_1 \wedge \neg v_2 \wedge v_3 \wedge \cdots \wedge \neg v_m \vee$

$\ldots$

$\neg v_1 \wedge \neg v_2 \wedge \neg v_3 \wedge \cdots \wedge v_m$

Size: $\mathcal{O}(m^2)$

Boundary conditions

„The variables always describe a configuration“

$$R = \bigwedge_t G(\text{state}_{t1}, \ldots, \text{state}_{tk}) \wedge$$
$$G(\text{pos}_{t,-p(n)}, \ldots, \text{pos}_{tp(n)}) \wedge$$
$$\bigwedge_i G(\text{band}_{ti1}, \ldots, \text{band}_{ti\ell})$$

Size $\mathcal{O}\left(p(n)(k^2 + p(n)^2 + p(n)\ell^2)\right) = \mathcal{O}\left(p(n)^3\right)$

Initial condition

„The var. describe the initial configuration $(1)w$“

$$A = \text{state}_{01} \wedge \text{pos}_{01} \wedge \bigwedge_{j=1}^{n} \text{band}_{0jw_j} \wedge \bigwedge_{j=-p(n)}^{0} \text{band}_{0j\sqcup} \wedge \bigwedge_{j=n+1}^{p(n)} \text{band}_{0j\sqcup}$$

Total $\mathcal{O}(1+1+n+p(n)+p(n)-n) = \mathcal{O}(p(n))$

Transition conditions U_1, $t \rightarrow t+1$

„At the head position corresponded transition according δ“

$$U_1 = \bigwedge_{t,z,i,a} \Big((\text{state}_{tz} \wedge \text{pos}_{ti} \wedge \text{band}_{tia}) \rightarrow \bigvee_{\{(z',a',y) \in \delta(z,a)\}} (\text{state}_{t+1,z'} \wedge \text{pos}_{t+1,i+y} \wedge \text{band}_{t+1,ia'}) \Big)$$

whereby the head movements are interpreted as numbers,

$L = -1, N = 0, R = +1.$

Size $\mathcal{O}((p(n) \cdot k \cdot p(n) \cdot \ell) \cdot (k \cdot \ell \cdot 3)) = \mathcal{O}\left(p(n)^2\right)$

Transition conditions U_2, $t \rightarrow t+1$

„At the non head positions nothing is changing“

$$U_2 = \bigwedge_{t,i,a} ((\neg \text{pos}_{ti} \wedge \text{band}_{tia}) \rightarrow (\text{band}_{t+1,i,a}))$$

Size $\mathcal{O}((p(n) \cdot p(n) \cdot \ell)) = \mathcal{O}\left(p(n)^2\right) = \mathcal{O}\left(p(n)^2\right)$

End condition E

„M accepts w in $\leq p(n)$ steps“

$$E = \bigvee_{t \leq p(n)} \bigvee_{z \in H} \text{state}_{t,z}$$

Total $\mathcal{O}(p(n))$

Total size von F

R	Boundary condition	$\mathcal{O}\left(p(n)^3\right)$
A	Initial condition	$\mathcal{O}\left(p(n)\right)$
U_1	Transition condition 1	$\mathcal{O}\left(p(n)^2\right)$
U_2	Transition condition 2	$\mathcal{O}\left(p(n)^2\right)$
E	End condition „M accepts w in $\leq p(n)$ steps“	$\mathcal{O}\left(p(n)\right)$

Total $\mathcal{O}\left(p(n)^3\right)$.

This is again polynomial

and can in this time automatically written.

Proposition: $\mathbf{NP} \subseteq \bigcup_{p \text{ polynomial}} \text{TIME}(2^{\mathcal{O}(p(n))})$

Proof:

Let $T = (Q, \Sigma, \Gamma, \delta, s, F)$ be NTM, that recognized L with time bound $p(n)$.

One get a deterministic algorithm, such that all combinations $\leq p(n)$ of nondeterministic choice systematically try to fulfill.

There are maximum

$$(3 \cdot |Q| \cdot |\Gamma|)^{p(n)} = 2^{\log_2(3 \cdot |Q| \cdot |\Gamma|) p(n)} = 2^{\mathcal{O}(p(n))}$$

such combinations.

Another NP-complete problems

Observation: $\leq_p$ is transitive, i.e.,

$\forall L, L', L'' : L \leq_p L' \wedge L' \leq_p L'' \longrightarrow L \leq_p L''$.

Lemma A: $L \in \mathbf{NP} \wedge SAT \leq_p L \longrightarrow L$ is **NP**-complete.

Proof: We have to prove $\forall L' \in \mathbf{NP} : L' \leq_p L$.

SAT is **NP**-complete also $L' \leq_p$SAT.

By the associativity of $\leq_p$ it follows that

$L' \leq_p$ SAT $\leq_p L$.

qed

Another NP-complete problems

Lemma A: $L \in \mathbf{NP} \wedge SAT \leq_p L \longrightarrow L$ is **NP**-complete.

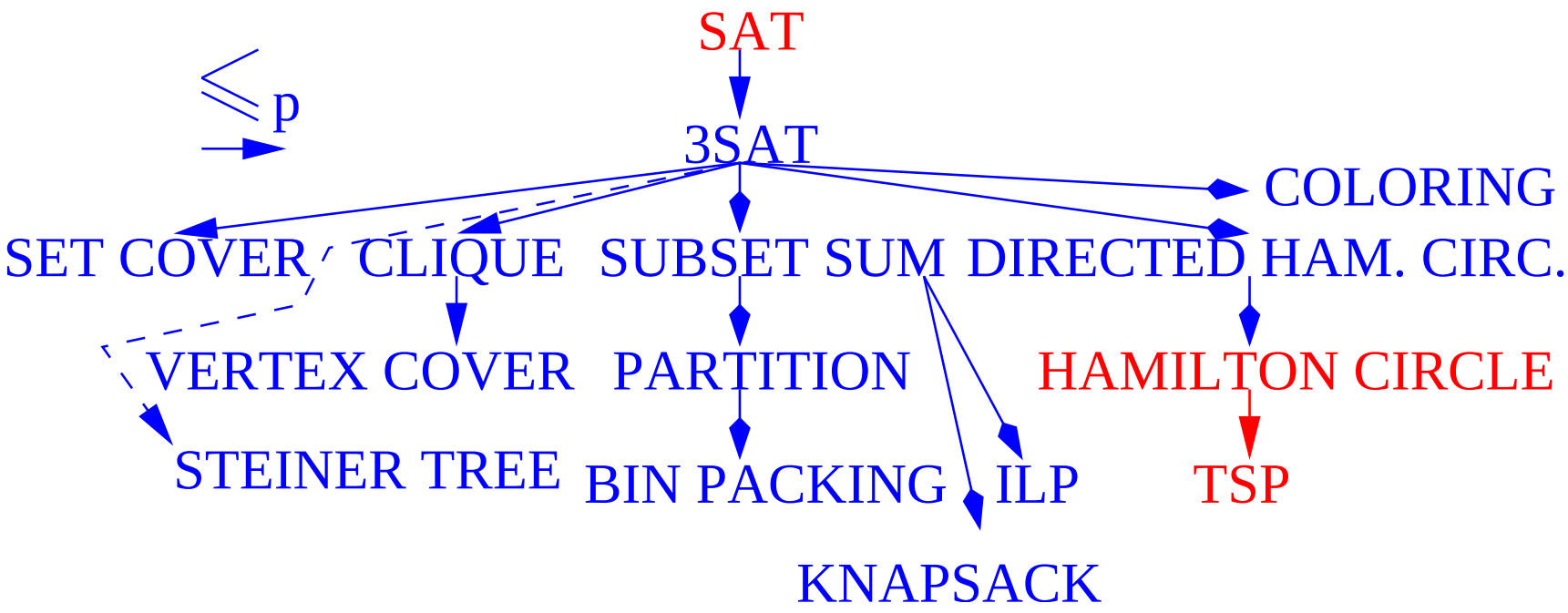
Lemma A':

$L \in \mathbf{NP} \wedge L'$ is NP complete $\wedge L' \leq_p L$

$\longrightarrow L$ is **NP**-complete.

Proof: Analog of the proof of Lemma A.

Another NP-complete problems



3SAT (CNF)

Given: Boolean formula in conjunctive normal form, max. 3 Literals per Clause

Question: Is F satisfiable?

Example: $(x \vee \neg y \vee z)(\neg u \vee x \vee z)(\neg x)$

Variable: $x, y, z, \ldots$

Literal: Variable, $\neg$Variable

clause: Literal$\vee \cdots \vee$Literal

CNF formula: (clause)$\cdots$(clause)

Proposition: 3SAT is NP-complete

Proof:

Since SAT is in **NP**, it is enough to show that 3SAT is **NP**-hard and

SAT$\leq_p$3SAT

Case: It does not help to bring in CNF at all.

- CNF could be in exponential size
- One necessarily does not get 3CNF

Proof „3SAT is NP-hard“

We develop a polynomial Alg.,

such that it transforms a 3CNF-formula F in satisfiable equivalent formula F', i.e.,

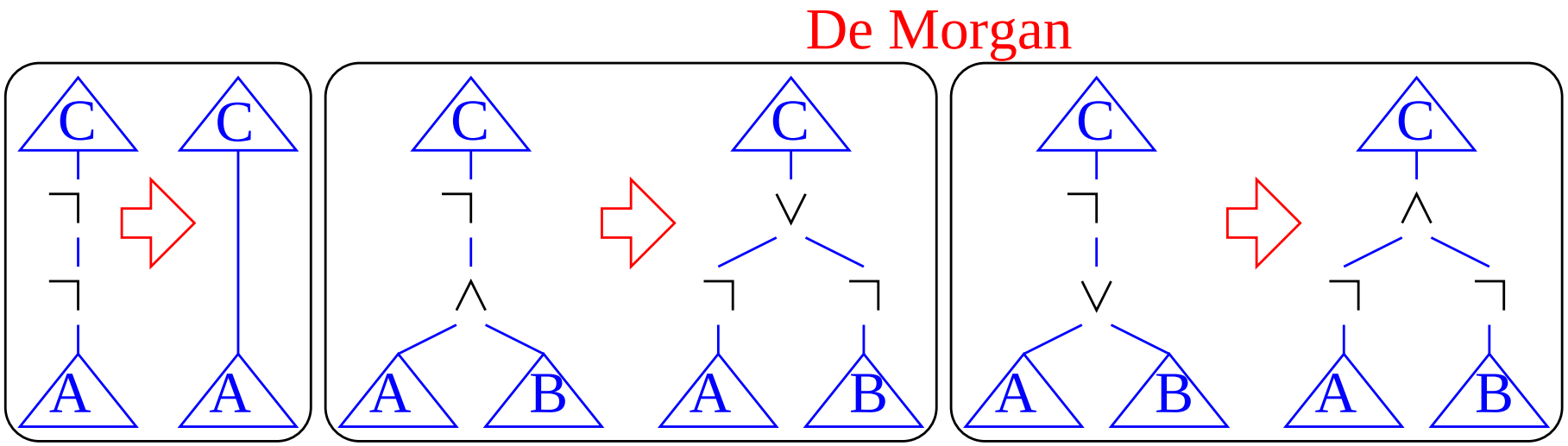
$$F \text{ satisfiable} \Leftrightarrow F' \text{ satisfiable}$$

Basic approach: a sequence of satisfiable equivalent transformations.

We consider the formula as a tree with max. grad two.

Leaves: variables x (or Literal $\overline{x} = \neg x$) Another Knots: $\wedge$, $\vee$, $\neg$

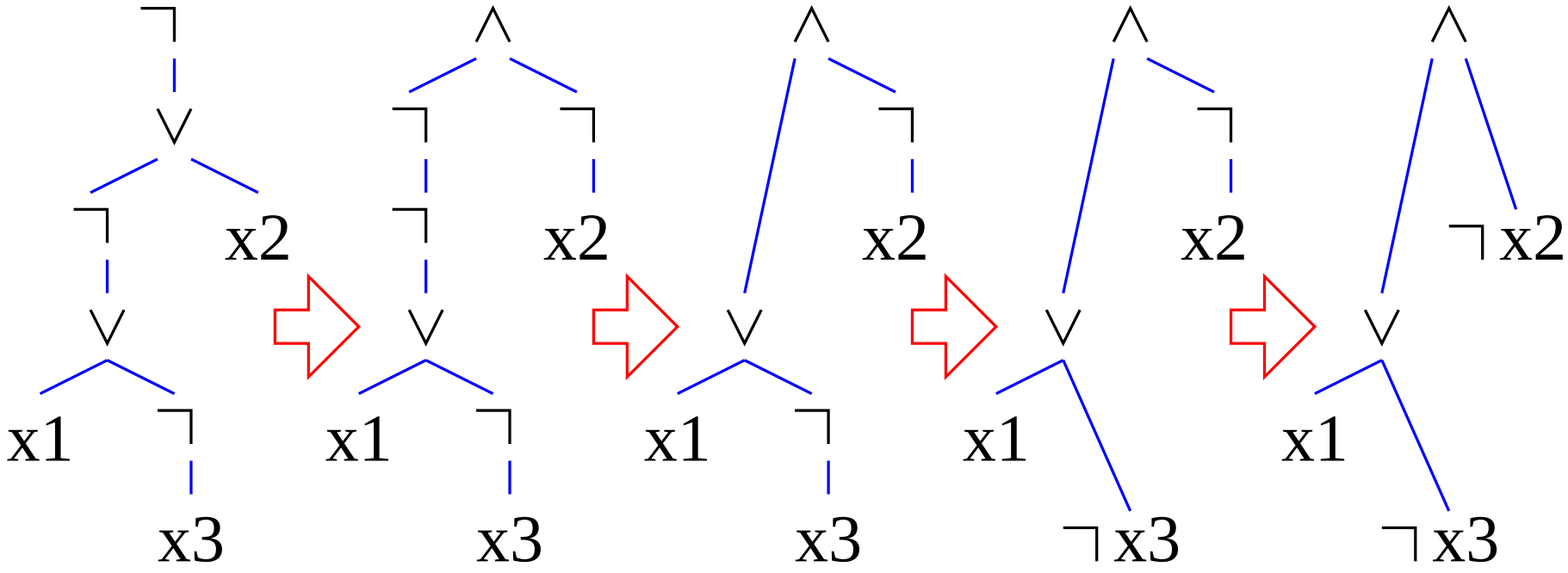
Putting the negations in the leaves



By traversing from up to down runs in linear time (RAM).

postcondition: formula letting as binary tree with inner knots $\vee$, $\wedge$ and Literals as leaves interpreting.

Example



Putting the negations on the leaves

normalize$(\neg x)$ **return** negNormalize(x)

normalize$(x \wedge y)$ **return** normalize$(x) \wedge$ normalize(y)

normalize$(x \vee y)$ **return** normalize$(x) \vee$ normalize(y)

normalize(v) **return** v

negNormalize$(\neg x)$ **return** normalize(x)

negNormalize$(x \wedge y)$ **return** negNormalize$(x) \vee$ negNormalize(y)

negNormalize$(x \vee y)$ **return** negNormalize$(x) \wedge$ negNormalize(y)

negNormalize(v) **return** $\neg v$

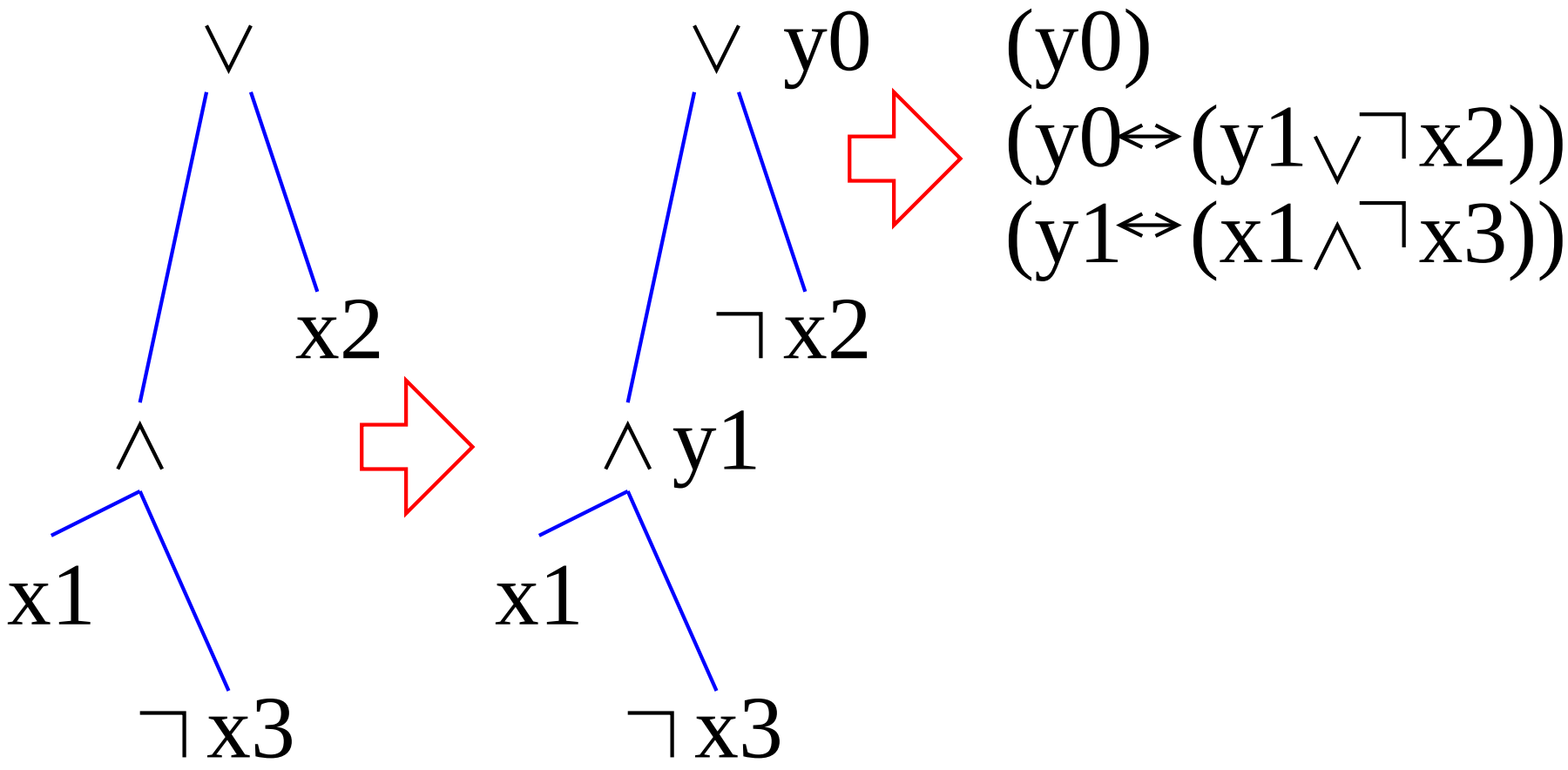
Non leaves knots $\longrightarrow$ New variables

Arrange new variables for $\wedge$, $\vee$.

Let y_0 be the literal on the root.

$$F_1 := (y_0) \wedge \Big(\bigwedge_{\overbrace{y \wedge z}^{v}} v \leftrightarrow (y \wedge z) \Big) \wedge \Big(\bigwedge_{\overbrace{y \vee z}^{v}} v \leftrightarrow (y \vee z) \Big)$$

Example



Equivalence by satisfiability for F_1

Proof F is satisfiable $\longrightarrow$ F_1 is satisfiable:

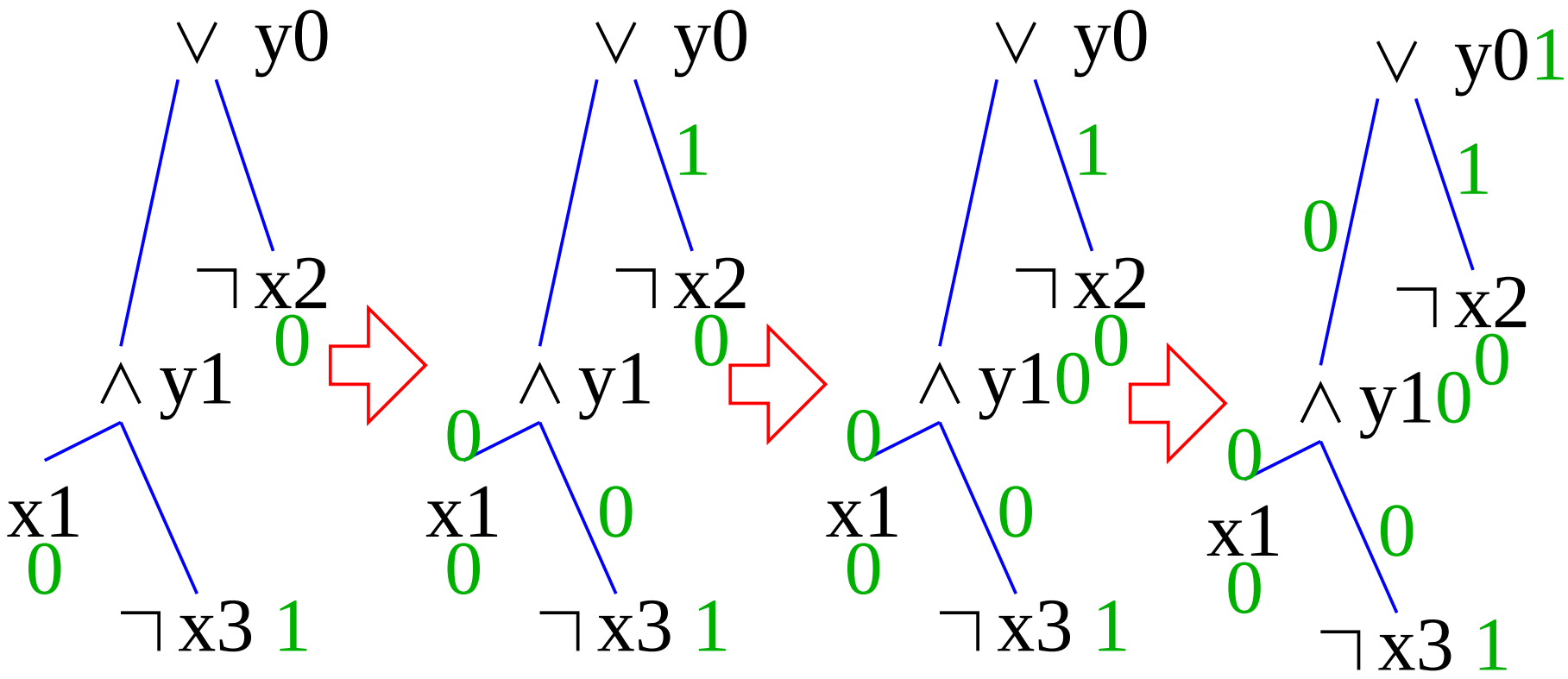
Consider satisfiable assignment for F.

Take over these as partial assignment for F_1.

Evaluate the formula from bottom up.

Take over the resulting truth values of the subtree with the root variable v for the assignment von v.

Example


∨ y0
¬ x2
0
∧ y1
x1
0
¬ x3 1
∨ y0
1
¬ x2
0
∧ y1
0
x1
0
0
¬ x3 1
∨ y0
1
¬ x2
0
∧ y1
0
0
x1
0
0
¬ x3 1
∨ y01
1
0
¬ x2
0
∧ y10
0
x1
0
0
¬ x3 1

Equivalence by satisfiability for F_1

Proof F_1 is satisfiable $\longrightarrow$ F is satisfiable:

Consider the assignment for F_1.

Take over the truth values of the leaves for F.

Non leaves knots for F have the same truth value as the corresponding variables in F_1.

y_0 is by 1 evaluated.

F is finally true.

$F_1 \rightsquigarrow$ 3CNF

Build CNF for each subformula:

$$a \leftrightarrow (b \vee c) \rightsquigarrow (a \vee \neg b)(\neg a \vee b \vee c)(a \vee \neg c)$$

$$a \leftrightarrow (b \wedge c) \rightsquigarrow (\neg a \vee b)(\neg a \vee c)(a \vee \neg b \vee \neg c)$$

a	b	c	$b \vee c$	$b \wedge c$	$a \leftrightarrow (b \vee c)$	$a \leftrightarrow (b \wedge c)$
0	0	0	0	0	1	1
0	0	1	1	0	0	1
0	1	0	1	0	0	1
0	1	1	1	1	0	0
1	0	0	0	0	0	0
1	0	1	1	0	1	0
1	1	0	1	0	1	0
1	1	1	1	1	1	1

excursus: 2SAT$\in$ P

Given : Boolean formula in conjuctive normal form, max. 2 literal per clause.

Question: Is F satisfiable?

Define $\overline{x}:= \neg x$, $\neg\overline{x}:= x$.

graph $G = (V,E)$, $V:= \{x, \neg x : x \in F\}$,

$E:= \{(\overline{A},B),(\overline{B},A) : (A \vee B) \in F\}$

Observation: F satisfiable

$\neg\exists$ cycle C in G, variable $x : x \in C \wedge \neg x \in C$.

These conditions we can verify in linear time.

(Strong connected component.)

SET COVER

Given:

Basic set M

Set system $\mathscr{T} \subseteq 2^M$

Parameter n

Example:

$\{1,2,3,4,5\}$

$\{\{1,3\},\{1,2\},\{2,4,5\}\}$

2

Question:

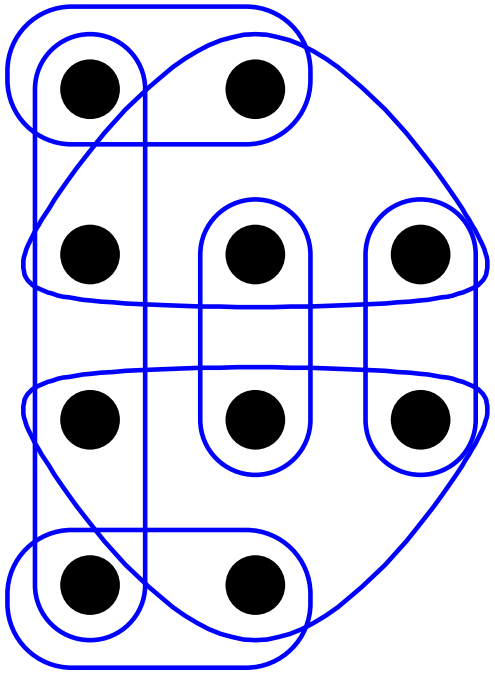
Are there elements $T_1 \in \mathscr{T}, \ldots, T_n \in \mathscr{T}$?

$T_1 \cup \cdots \cup T_n = M$?

Yes:

$\{1,3\},\{2,4,5\}$

Set Cover Example



The minimal n: 3

Proposition: SET COVER is NP-complete

Proof of SET COVER $\in$ **NP**:

Select nondet. n sets $T_1 \in \mathcal{T}, \ldots, T_n \in \mathcal{T}$

Construct the union $M' = T_1 \cup \cdots \cup T_n$

Check if $M = M'$

Proof „SET COVER is NP-hard“

We show 3SAT$\leq_p$SET COVER.

Let $F = K_1 \wedge \cdots \wedge K_m$ one CNF-formula with variables $x_1, \ldots, x_n$.

Select $M = \{1, \ldots, m+n\}$.

$T_i := \{ j : x_i \text{ occurs in clause } K_j \} \cup \{m+i\}$

$T_i' := \{ j : \neg x_i \text{ occurs in clause } K_j \} \cup \{m+i\}$

Set $\mathcal{T} := \{T_1, \ldots, T_n, T_1', \ldots, T_n'\}$.

Parameter n.

To show: The Set Cover instance $(M, \mathcal{T}, n)$ is solvable if F satisfiable.

Example

$F = (x_1 \vee x_2 \vee x_3)(x_2 \vee \neg x_3 \vee x_4)(\neg x_3 \vee \neg x_4 \vee \neg x_1)$

$n = 4, m = 3.$

$T_1 = \{1,4\},$ $\{3,4\} = T_1'$

$T_2 = \{1,2,5\},$ $\{5\} = T_2'$

$T_3 = \{1,6\},$ $\{2,3,6\} = T_3'$

$T_4 = \{2,7\},$ $\{3,7\} = T_4'$

$T_1 \cup T_2 \cup T_3' \cup T_4 = \{1,2,3,4,5,6,7\}$

So $x_1 = x_2 = x_4 = 1, x_3 = 0$

Proof F satisfiable $\longrightarrow (M, \mathscr{T}, n)$ is solvable.

Select $M = \{1, \ldots, m+n\}$.

T_i:= $\{ j : x_i$ occurs in clause $K_j \} \cup \{m+i\}$

T_i':= $\{ j : \neg x_i$ occurs in clause $K_j \} \cup \{m+i\}$

for $i = 1..n$

Select T_i in case of $x_i = 1$.

Select T_i' in case of $x_i = 0$.

$\{m+1, \ldots, m+n\}$ will be covered,

since for each variable x_i either T_i or T_i' will be selected.

$j \in \{1, \ldots, m\}$ will be covered:

Let L be fulfilled literal in K_j

Case $L = x_i \longrightarrow x_i = 1 \longrightarrow j \in T_i$ and T_i will be selected.

Case $L = \neg x_i \longrightarrow x_i = 0 \longrightarrow j \in T_i'$ and T_i' will be selected.

Proof $(M, \mathscr{T}, n)$ solvable $\longrightarrow$ F satisfiable

$M = \{1, \ldots, m+n\}$.

T_i:= $\{ j : x_i$ occurs in clause $K_j \} \cup \{m+i\}$

T_i':= $\{ j : \neg x_i$ occurs in clause $K_j \} \cup \{m+i\}$

$\{m+1, \ldots, m+n\}$ will be covered $\longrightarrow$

for every variable x_i either T_i or T_i' will be selected.

$T_i \rightsquigarrow$ set $x_i = 1$.

$T_i' \rightsquigarrow$ set $x_i = 0$.

$\forall j \in 1..m$:

Case 1, $j \in T_i$, T_i will be selected: $x_i \in K_j$ makes K_j true $\quad (x_i = 1)$

Case 1, $j \in T_i'$, T_i' will be selected: $\neg x_i \in K_j$ makes K_j true $\quad (x_i = 0)$

CLIQUE

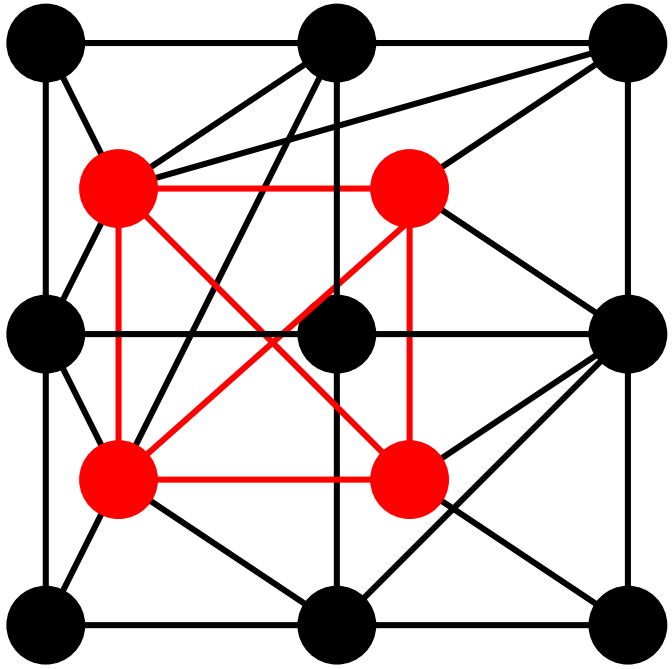
Given: A undirected graph $G = (V, E)$,

Parameter $k \in \mathbb{N}$.

Question:

Does G contain a clique of size k?

i.e. $\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v \in V' : \{u, v\} \in E$



Proof Clique$\in$ NP

Guess k knots from V'

Check if they build a clique.

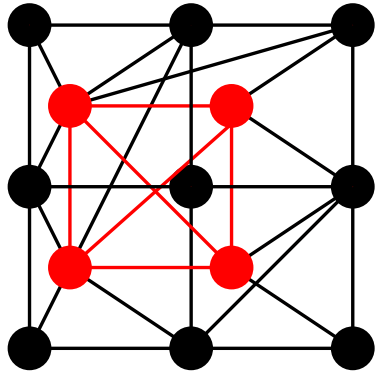
Given: Undirected graph $G = (V, E)$,

Parameter $k \in \mathbb{N}$.

Question:

Does G contain one clique of size k?

i.e. $\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v \in V' : \{u, v\} \in E$



Proof for „CLIQUE is NP-hard“

We show 3SAT$\leq_p$CLIQUE.

Let $F = (z_{11} \vee z_{12} \vee z_{13}) \wedge \cdots \wedge (z_{m1} \vee z_{m2} \vee z_{m3})$ with literals $z_{ij} \in \{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots \neg x_n\}$.

Construct a graph $G = (V, E)$:

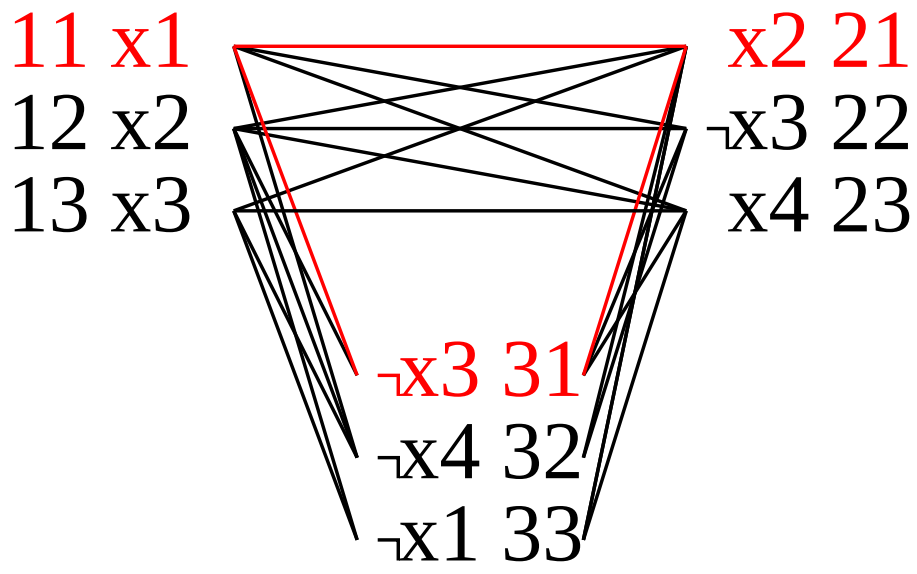
$V = \{(1,1), (1,2), (1,3), \ldots, (m,1), (m,2), (m,3)\}$,

$E = \{\{(i,j), (p,q)\} : i \neq p, z_{ij} \neq \overline{z_{pq}}\}$.

Parameter $k = m$.

Example:

$$F = (x_1 \vee x_2 \vee x_3)(x_2 \vee \neg x_3 \vee x_4)(\neg x_3 \vee \neg x_4 \vee \neg x_1)$$



Idea: since in each clause there is a fulfilled literal.

The edges connect „compatible“ literals

$$F = (z_{11} \vee z_{12} \vee z_{13}) \wedge \cdots \wedge (z_{m1} \vee z_{m2} \vee z_{m3}).$$

$$V = \{(1,1),(1,2),(1,3),\ldots,(m,1),(m,2),(m,3)\},$$

$$E = \left\{\{(i,j),(p,q)\} : i \neq p, z_{ij} \neq \overline{z_{pq}}\right\}.$$

F satisfiable $\longrightarrow$ $G = (V,E)$ has m-clique

F satisfiable by assignment $B : \{x_1,\ldots,x_n\} \to \{0,1\}$

$\longrightarrow \forall j \in 1..m : \exists z_{j,k_j} : B$ makes z_{jk_j} true

These literals are pairwise not complement each other

and comes from different clauses.

$\longrightarrow (1,k_1),\ldots,(m,k_m)$ are pairwise connected.

We build also one m-clique

qed

$$F = (z_{11} \vee z_{12} \vee z_{13}) \wedge \cdots \wedge (z_{m1} \vee z_{m2} \vee z_{m3}).$$

$$V = \{(1,1),(1,2),(1,3),\ldots,(m,1),(m,2),(m,3)\},$$

$$E = \left\{\{(i,j),(p,q)\} : i \neq p, z_{ij} \neq \overline{z_{pq}}\right\}.$$

$G = (V,E)$ has m-clique $\longrightarrow$ F satisfiable

$(j_1,k_1),\ldots(j_m,k_m)$ is m-clique with literals z_{j_i,k_i}.

Only edges between different j_i.

$\longrightarrow$ these j_i are pairwise different.

$\longrightarrow j_1,\ldots,j_m = 1,\ldots,m$

Only edges between non complemented literals.

$\longrightarrow$ these literals are pairwise non complemented.

$\longrightarrow \exists$ assignment $B : \forall j \in 1..m : z_{jk_j}[B] = 1$

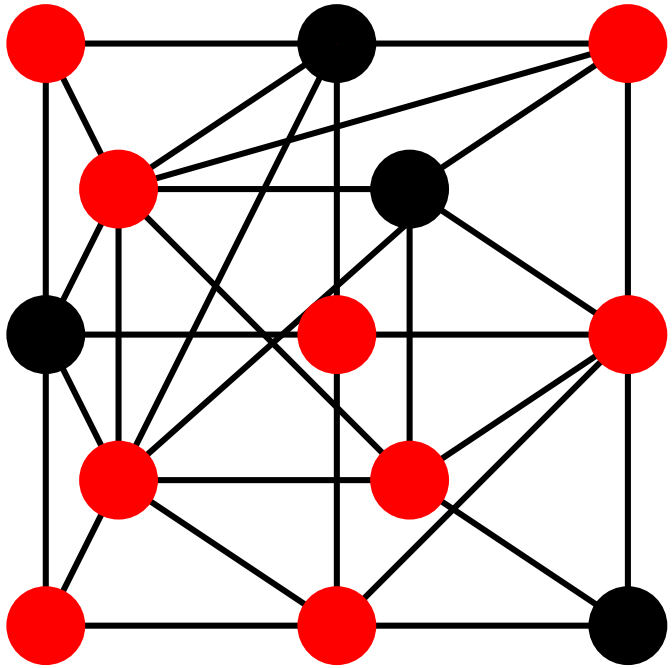
$\longrightarrow F$ satisfiable by B qed

VERTEX COVER (knots covered)

Given: undirected graph $G = (V, E)$,

Parameter $k \in \mathbb{N}$.

Question: $\exists V' \subseteq V : |V'| = k \wedge \forall \{u, v\} \in E : u \in V' \vee v \in V'$



Proof VERTEX COVER $\in$ NP

. . . as the previous. . .

Proof von „VERTEX COVER is NP-hard“

We show CLIQUE$\leq_p$VERTEX COVER.

Consider the complemented graph
$\overline{G} = (V, \overline{E})$ with $\overline{E} = \{u \neq v \in V : \{u,v\} \notin E\}$.

G has Vertex Cover V' of size k $\Leftrightarrow$

$\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v : \{u,v\} \in E \rightarrow (u \in V' \vee v \in V')$ $\Leftrightarrow$

$\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v : \{u,v\} \notin E \vee (u \in V' \vee v \in V')$ $\Leftrightarrow$

$\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v : \{u,v\} \notin E \vee \neg(u \notin V' \wedge v \notin V')$ $\Leftrightarrow$

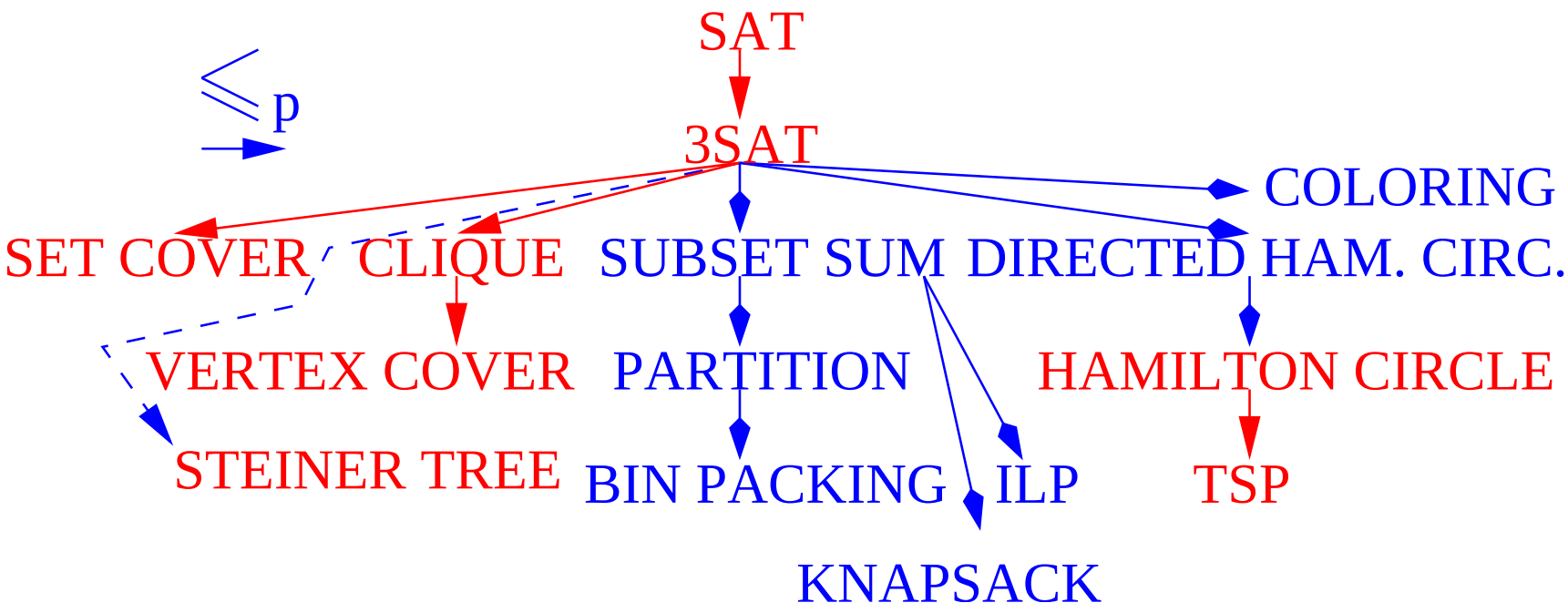
$\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v : (u \notin V' \wedge v \notin V') \rightarrow \{u,v\} \notin E$ $\Leftrightarrow$

$\exists V' \subseteq V : |V'| = k \wedge \forall u \neq v : \{u,v\} \in V \setminus V' \rightarrow \{u,v\} \in \overline{E}$ $\Leftrightarrow$

$\exists \overline{V}' \subseteq V : |\overline{V}'| = |V| - k \wedge \forall u \neq v : \{u,v\} \in \overline{V}' \rightarrow \{u,v\} \in \overline{E}$ $\Leftrightarrow$

$\overline{G}$ has one clique of size $|V| - k$

Viewed Reductions



Proof technics for $A \leq_p B$:

Special instance

Show, that A is a special instance (in a wider sense) for B.

Every elem. object in the formulated problem for A corresponds to one elementary object in the formulated problem in B.

Example: HAMILTON CYCLE$\leq_p$TSP.

Example: SUBSET SUM$\leq_p$KNAPSACK.

Proof technics for $A \leq_p B$:

Another interpretation

Show, that A is a special instance (in a wider sense) for B.

Every elementary object in the formulated problem for A corresponds to one elementary object in the formulated problem in B.

Example: VERTEX COVER$\leq_p$CLIQUE.

Example: CLIQUE$\leq_p$VERTEX COVER.

Proof technics for $A \leq_p B$:

Gadgets

Every elementary object in the formulated problem for A will construct more connected objects in B .

Example: 3SAT$\leq_p$SET COVER; $x_i \rightsquigarrow T_i, T_i'$.

Example: SAT$\leq_p$3SAT; knots for $F \rightsquigarrow$ three clauses.

Example: $L \leq_p$SAT;

- States, head position, tape symbols $\rightsquigarrow$ many variables.
- Initial configuration $\rightsquigarrow$ formula A
- $\delta \rightsquigarrow$ transition fit U_1
- Final states$\rightsquigarrow$ End condition E

Proof technics for $A \leq_p B$:

Border conditions

Force certain characteristics by additional construction.

Example: $L \leq_p$SAT; R, U_2.

Example: SAT$\leq_p$3SAT; (y_0).

Example: 3SAT$\leq_p$COLORING

COLORING

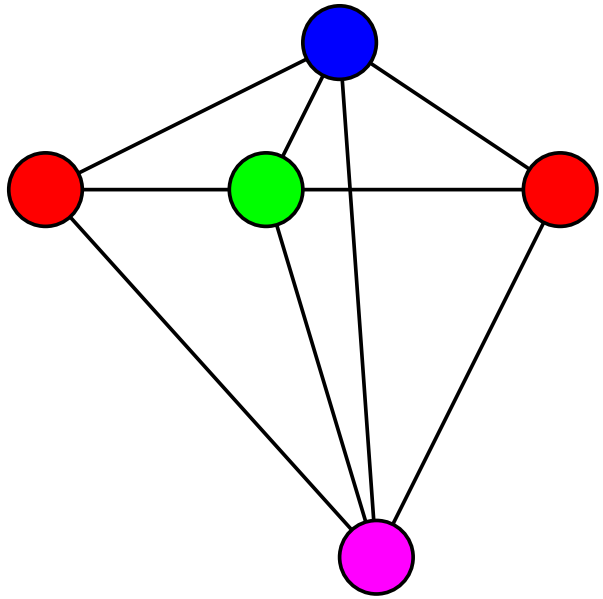
Given: undirected graph $G = (V, E)$,

Parameter $k \in \mathbb{N}$.

Question:

Is there one knots coloring $c : V \to \{1, \ldots, k\}$

with $\forall \{u, v\} \in E : c(u) \neq c(v)$.



Applications of coloring

- Maps coloring (4 are enough !)
- The set of knots in independent sets factoring.

  For example for parallelization.
- Register allocation.

  knots=In one procedure compute the value,

  edges=The value could not be in the same register.
- ...

Related problem: Frequency of a radio sender

Proof COLORING $\in$ NP

. . . the same. . .

Proof von „COLORING is NP-hard“

We show 3SAT$\leq_p$3-COLORING (i.e. $k=3$).

Let $F = K_1 \wedge \cdots \wedge K_m$ with literals from

$\{x_1, \ldots x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}$.

Exactly 3 literals per clause.

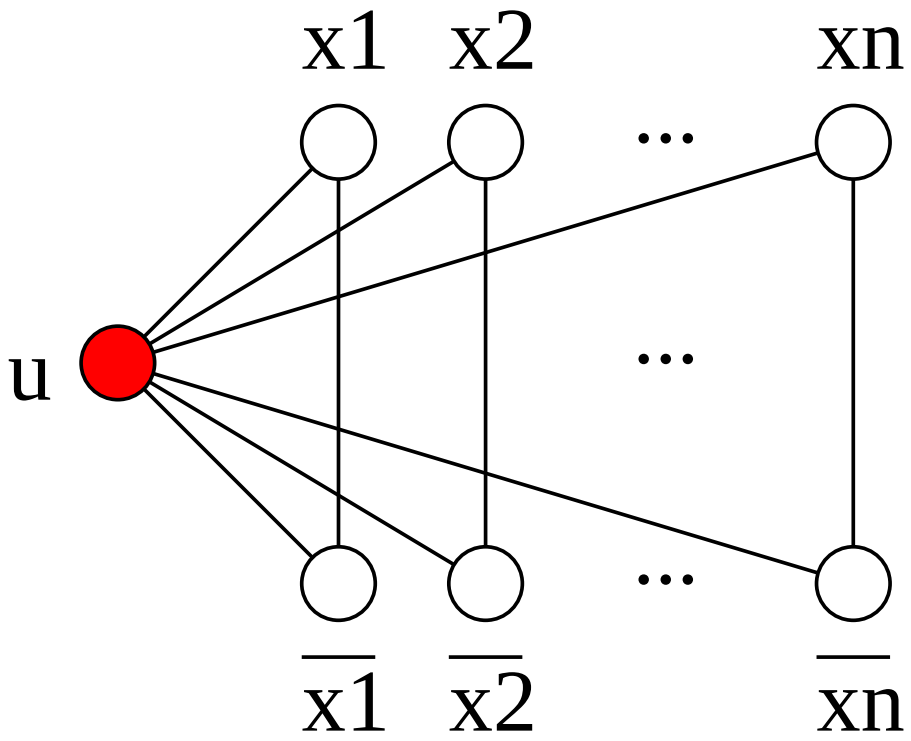
Idea:

Construct a graph $G = (V, E)$ with $2n + 5m + 2$ knots, so that

G 3-coloring $\Leftrightarrow$ F satisfiable.

Boundary condition forces consistent truth values for literals

One part of G:



Colors $0 =$ true, $1 =$ false, $2 =$ rot $= c(u)$.

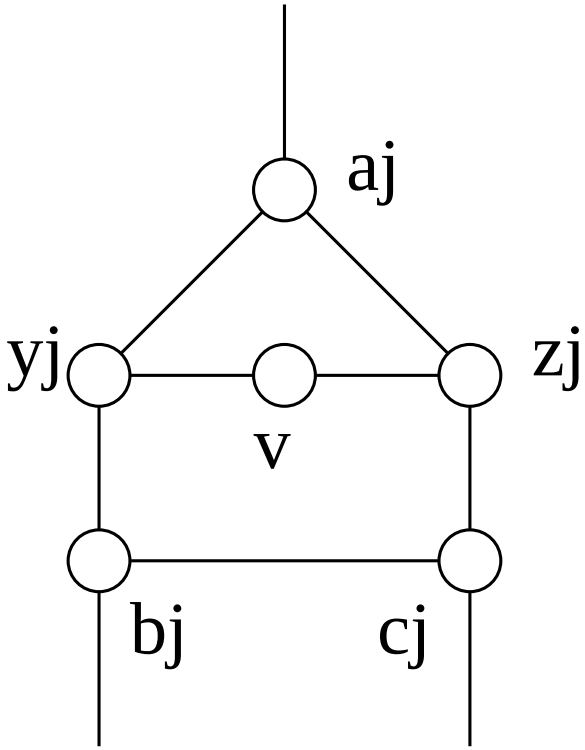
Gadget for clause $K_j = (a_j \vee b_j \vee c_j)$

v: All over equal $(u,v) \in E$

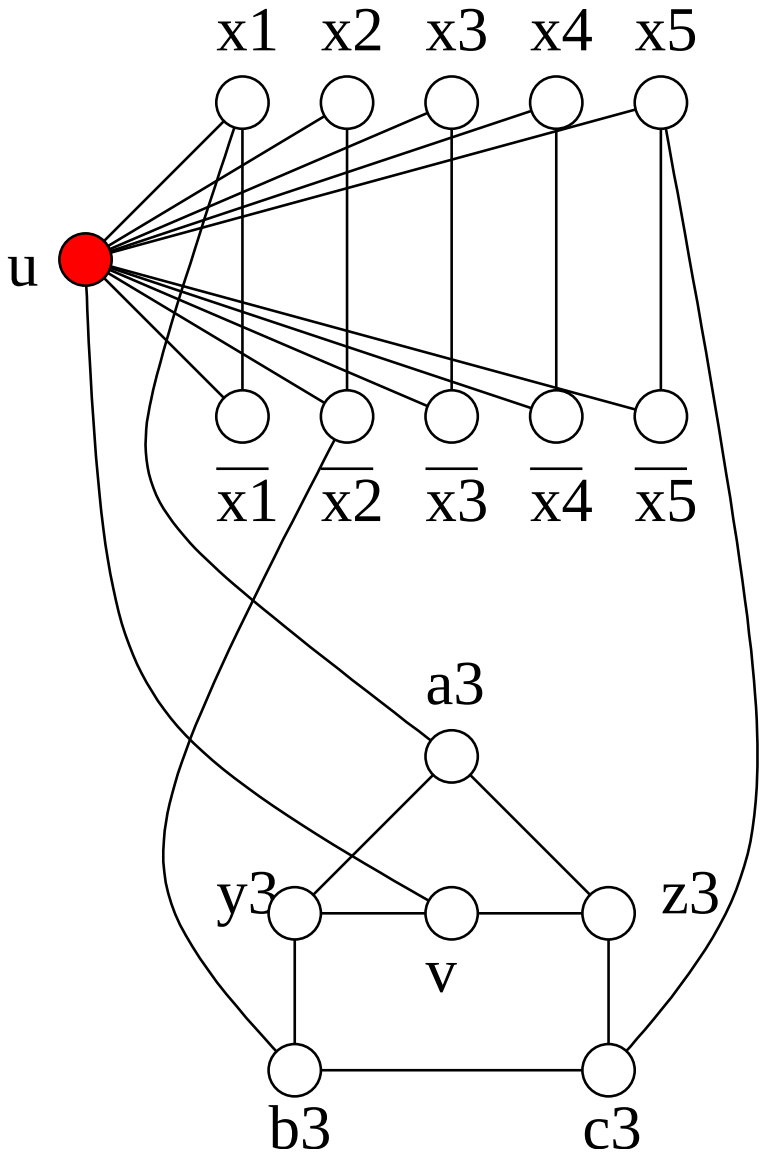
a_j, b_j, c_j: Own knots for every clause.

Edges correspond to literal knots.

Attention: Color $a_j \neq$Truth value a_j,...!



Example: Gadget for clause $K_3 = (x_1 \vee \overline{x_2} \vee x_5)$

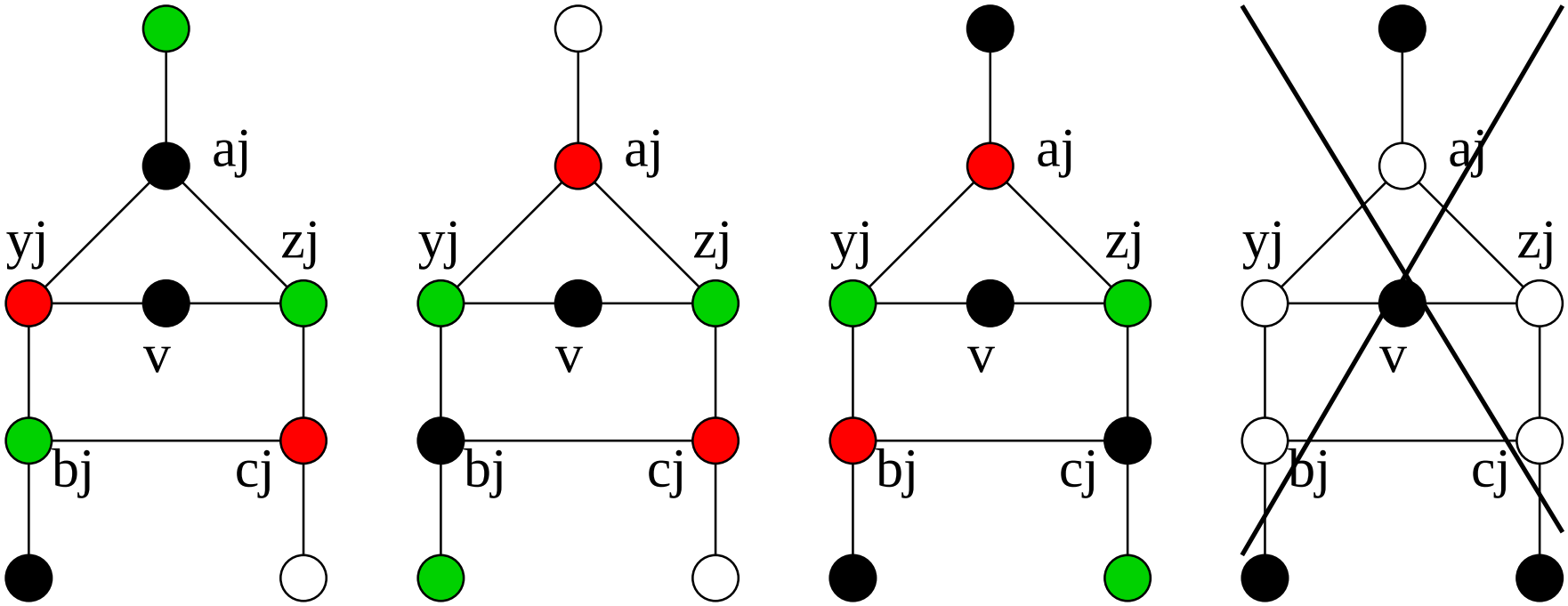


Proof: F satisfiable $\longrightarrow$ G 3-coloring

$c(u)$:= rot, $c(v)$:= false,

select $c(x_i) = \neg c(\overline{x_i})$ corresponded to one satisfiable assignment.

Plainly case distinction for clause-Gadget j:



Proof: G 3-coloring $\longrightarrow$ F satisfiable

Name the color von u rot.

Name the color $c(v) \neq c(u)$ false.

Name the 3. color true.

The graph forces

consistent truth values for literal knots $x_i, \overline{x_i} \in \{\text{true}, \text{false}\}$.

To show:

The assignment B makes F true— $F[B] = \text{true}$

Proof by contradiction.

Assume $F[B] = \text{false}$?

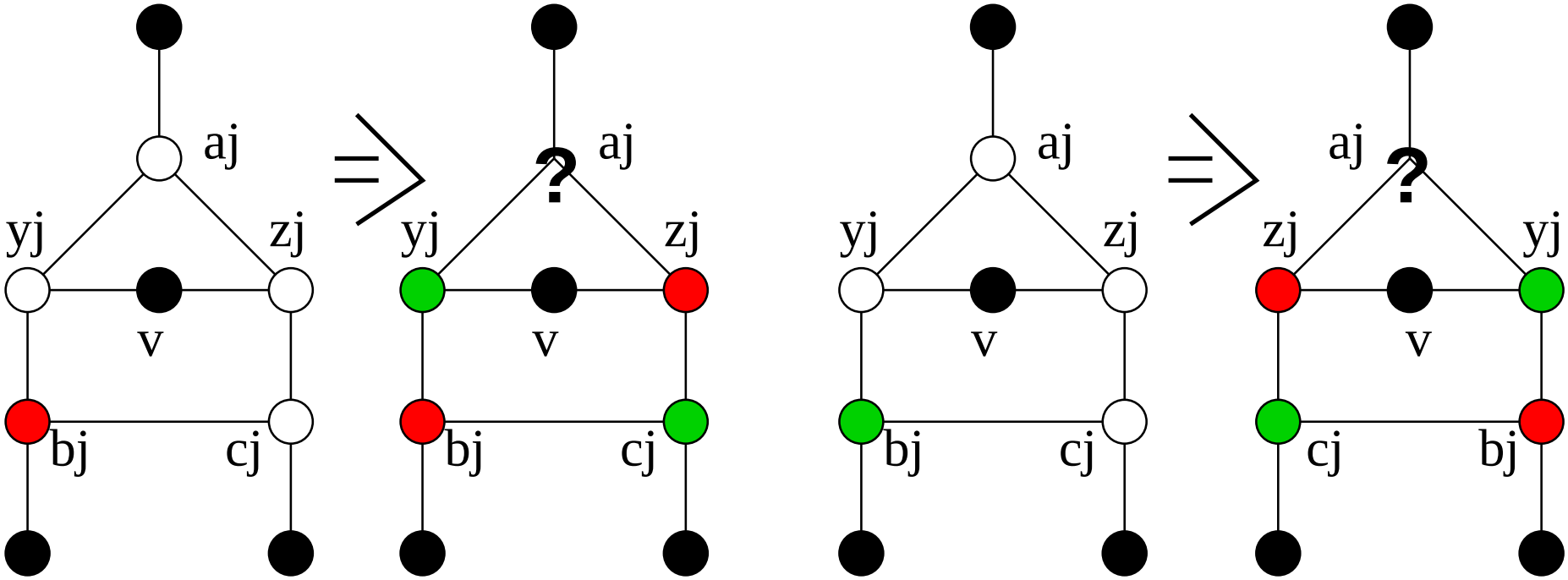
$\longrightarrow \exists j$: $K_j[B] = \text{false}$.

$\longrightarrow$ All literals from K_j are false.

Proof: G 3-coloring $\longrightarrow$ F satisfiable

$[\cdots]$ All literals from K_j are false

$\longrightarrow c(b_j) =$rot oder $c(b_j) =$true



Evidently $F[B]$ is also true.

qed

Excursus: 2-coloring$\in$ P

2-coloring graphs are called bipartit.

Observation: tree are 2-coloring.

Observation: The odd cycles are not 2-coloring. The sane is true for graphs, which have an odd cycle as a subgraph.

Excursus: 2-coloring $\in$ P

Consider an arbitrary strong connected component $C \subseteq V$.

Consider an arbitrary spanning tree T for C.

Consider 2-coloring c von T.

$\exists \{u,v\} \in E : c(u) = c(v)$?

$\longrightarrow \{u,v\} \cup \{$ Path from u to v in $T\}$ build odd cycle (*).

$\longrightarrow G$ is not a 2-coloring.

Then: c is legal 2-coloring for G.

(*):Exercise: Show by induction on the path size, that two knots with equal colors in T has even distance.

SUBSET SUM

Given: n Materials with weight $w_i \in \mathbb{N}$

Parameter $W \in \mathbb{N}$.

Question:

Is there a subset M von $\{1,\ldots,n\}$,

so that $\sum_{i \in M} w_i = W$

$\approx$Special case of knapsack with equal profits.

Attention! [Schöning] named SUBSET SUM RUCKSACK.

Schöning	Sanders
3CNF-SAT	3SAT
MENGENUBERDECKUNG	SET COVERING
KNOTENUBERDECKUNG	VERTEX COVER
RUCKSACK	SUBSET SUM
—	KNAPSACK (RUCKSACK)
FÄRBBARKEIT	COLORING

Proof SUBSET SUM $\in$ NP

. . . we know. . .

Proof „SUBSET SUM is NP-hard“

We show 3SAT$\leq_p$SUBSET SUM.

Let $F = K_1 \wedge \cdots \wedge K_m = (z_{11} \vee z_{12} \vee z_{13}) \wedge \cdots \wedge (z_{m1} \vee z_{m2} \vee z_{m3})$

with

literals $z_{ij} \in \{x_1, \ldots, x_n\} \cup \{\neg x_1, \ldots, \neg x_n\}$.

We define a SUBSET SUM instance with weights

$(v_1, \ldots, v_n, v'_1, \ldots v'_n, c_1, \ldots, c_m, d_1, \ldots, d_m)$

and sum W

The SUBSET SUM instance

$(v_1, \ldots, v_n, v'_1, \ldots v'_n, c_1, \ldots, c_m, d_1, \ldots, d_m), W$

$m+n$ decimal numbers (words in $\{0, \ldots, 9\}$)

Let v_{ij}:= the number that occurs for x_i in clause j. (Decimal digit)

Let v'_{ij}:= the number occurs for $\neg x_i$ in clause j. (Decimal digit)

$v_i = v_{i1} v_{i2} \cdots v_{im} 0^{i-1} 10^{n-i}$,

$v'_i = v'_{i1} v'_{i2} \cdots v'_{im} 0^{i-1} 10^{n-i}$

$c_j := 0^{j-1} 10^{m-j} 0^n$

$d_j := 0^{j-1} 20^{m-j} 0^n$

$W := \underbrace{4 \cdots 4}_{m} \underbrace{1 \cdots 1}_{n}$

Example

$F = (x_1 \vee \neg x_3 \vee x_5) \wedge (\neg x_1 \vee x_5 \vee x_4) \wedge (\neg x_2 \vee \neg x_2 \vee \neg x_5)$

$m = 3, n = 5$

v_1	$= 100\ 10000$	v'_1	$= 010\ 10000$
v_2	$= 000\ 01000$	v'_2	$= 002\ 01000$
v_3	$= 000\ 00100$	v'_3	$= 100\ 00100$
v_4	$= 010\ 00010$	v'_4	$= 000\ 00010$
v_5	$= 110\ 00001$	v'_5	$= 001\ 00001$
c_1	$= 100\ 00000$	d_1	$= 200\ 00000$
c_2	$= 010\ 00000$	d_1	$= 020\ 00000$
c_3	$= 001\ 00000$	d_1	$= 002\ 00000$

$W = 444\ 11111$

Proof idea

Observation:

by decimal addition the weights are tread no overflow.

$$W = 444\,11111$$

The right block could only rise, so for each variable x_i either$x_i \equiv v_i$ or $\neg x_i \equiv v_i'$ is selected.

Position j of the left block stays for clause j.

The selected v_i, $v_i' =$ numbers the fulfilled literals:

1,2 or 3 for fulfilled clauses.

The c_j, d_j are Slack-variables, the position j of fulfilled clauses of the value could fulfil 4 .

F satisfiable $\longrightarrow$ Instance solution

Consider assignment B with $F[B] = 1$.

Select for M, v_i when $x_i[B] = 1$ and v'_i when $x_i[B] = 0$.

Up to know it holds $\sum_{i \in M} w_i = t_1 t_2 \cdots t_m 1^n$

with $t_j \in \{1,2,3\}$ (each clause has 1–3 fulfilled literals).

for $j := 1$ **to** m **do**

- **if** $t_j = 1$ **then** select c_j and d_j // $1+1+2=4$
- **if** $t_j = 2$ **then** select d_j // $2+2=4$
- **if** $t_j = 3$ **then** select c_j // $3+1=4$

Now is $\sum_{i \in M} w_i = 4^m \; 1^n = W$.

qed.

Instance solution $\longrightarrow F$ satisfiable

Consider M with $\sum_{i \in M} w_i = W = 4^m\ 1^n$.

For each variable i will either v_i or v'_i fulfilled.

or else would be position i in right block $\neq 1$.

Select an assignment B with $x_i = 1$ if v_i fulfilled, $x_i = 0$ otherwise.

Statement: $F[B] = 1$ (By contradiction)

Assume: $F[B] = 0$ also $\exists j : K_j[B] = 0$.

$\longrightarrow \sum$ selected v_i, v'_i is 0 on position j

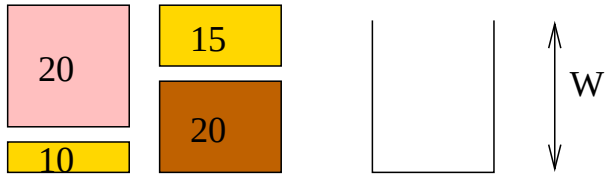
$\longrightarrow \sum$ selected c_i, d_i is $\leq 3 \neq 4$ on position j

A contradiction.

So $F[B] = 1$

qed.

Example Knapsack problem



- n objects with weights $w_i \in \mathbb{N}$ and profit p_i
- Select a subset $\mathbf{x}$ from the objects
- so that $\sum_{i \in \mathbf{x}} w_i \leq W$ and
- maximize the profit $\sum_{i \in \mathbf{x}} p_i$

Knapsack problem

$\in$ **NP**: we know

NP-hard: We show SUBSET SUM $\leq_p$ KNAPSACK

Obviously the SUBSET SUM is an instance

$(w_1, \ldots, w_n)$ with sum $= W$

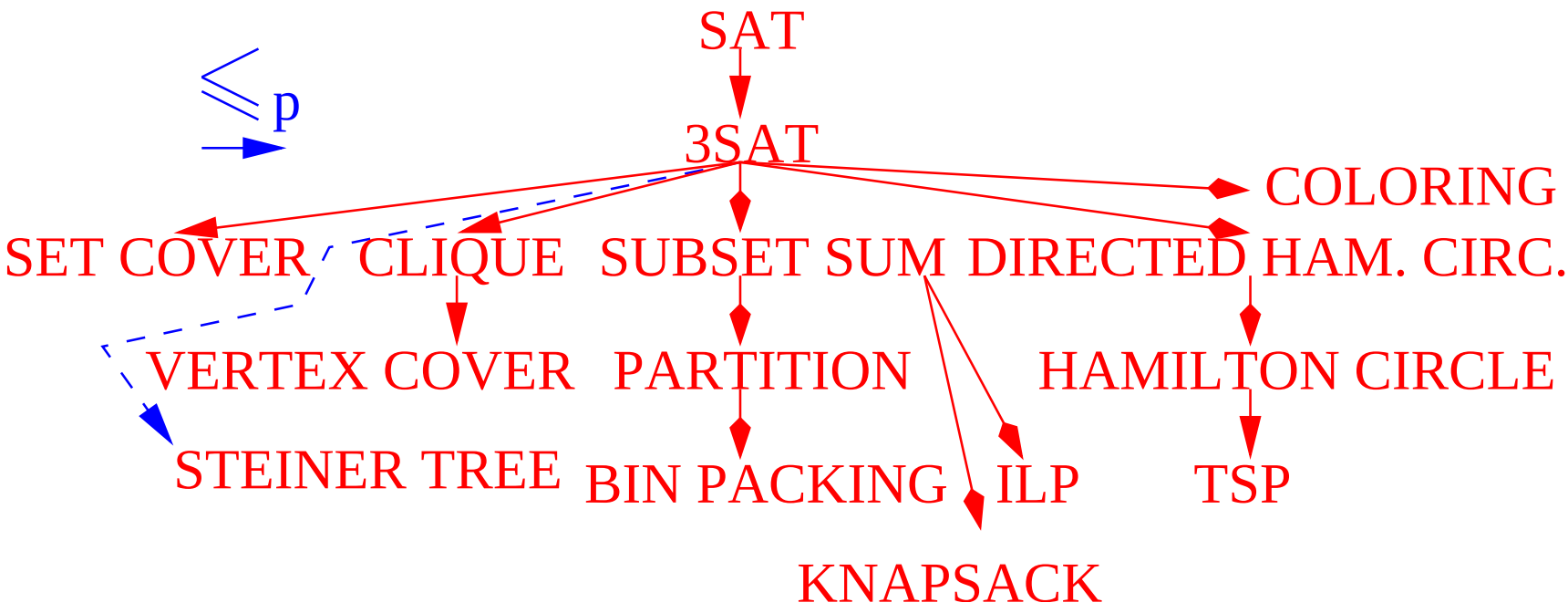
Equivalent for KNAPSACK-Instance

$\underbrace{(w_1, \ldots, w_n)}_{\text{weights}}, \underbrace{(w_1, \ldots, w_n)}_{\text{profit}}$ with

weight bound $\leq W$ and

profit bound $\geq W$.

Reductions


SAT
p
3SAT
COLORING
SET COVER
CLIQUE
SUBSET SUM
DIRECTED HAM. CIRC.
VERTEX COVER
PARTITION
HAMILTON CIRCLE
STEINER TREE
BIN PACKING
ILP
TSP
KNAPSACK

PARTITION

Given: n Objects with weights $w_i \in \mathbb{N}$

Question:

Is there a subset M von $\{1,\ldots,n\}$,

so that $\sum_{i\in M} w_i = \sum_{i\notin M} w_i$?

in **NP**, since ...

Proof SUBSET SUM$\leq_p$PARTITION

Given one SUBSET SUM Instance $S = (w_1, \ldots, w_k), W$.

Let $M := \Sigma_{i=1}^{k} w_i$.

Consider PARTITION Instance $P = (w_1, \ldots, w_k, M - W + 1, W + 1)$.

Total weight: $M + M - W + 1 + W + 1 = 2(M + 1)$.

to show P has solution $\Leftrightarrow$ S has solution.

SUBSET SUM Instance $S = (w_1, \ldots, w_k), W$. $M := \Sigma_{i=1}^{k} w_i$.

PARTITION Instance $P = (w_1, \ldots, w_k, M - W + 1, W + 1)$. $(\Sigma = 2(M+1))$.

Case P has solution $\longrightarrow$ S has solution:

Let J be a solution for P, i.e., $\sum_{j \in J} P[j] = M + 1$.

Either $M - W + 1$ or $W + 1$ will be selected:

$M - W + 1 + W + 1 = M + 2 > M + 1$, $\Sigma_{i=1}^{k} w_i = M < M + 1$

Case $M - W + 1$ will be selected:

Then it is $\sum_{j \in J \cap \{1,\ldots,k\}} P[j] = M + 1 - (M - W + 1) = W$.

Also it is $J \cap \{1, \ldots, k\}$ is solution for S.

Case $W + 1$ will be selected:

Then $\sum_{j \in J \cap \{1,\ldots,k\}} P[j] = M + 1 - (W + 1) = M - W$.

Also $\{1, \ldots, k\} \setminus J$ is solution for S.

SUBSET SUM Instance $S = (w_1, \ldots, w_k), W$.

$M := \Sigma_{i=1}^{k} w_i$.

PARTITION Instance $P = (w_1, \ldots, w_k, M - W + 1, W + 1)$.

Total weight: $2(M+1)$.

Case S has solution $\longrightarrow$ P has solution:

Let I be a solution for S, i.e., $\sum_{i \in I} w_i = W$.

Then $J := I \cup \{k+1\}$ is a solution for P, then

$$\sum_{j \in J} P[j] = \sum_{i \in I} w_i + M - W + 1 = W + M - W + 1 = M + 1$$

BIN PACKING

Given:

Dozen of cookies $b \in \mathbb{N}$.

Cookies of size $w_1,\ldots,w_n \in \mathbb{N}$

Number of cookies k.

Question:

Could we put all cookies somehow in the dozen?, i.e.,

$$\exists f : 1..n \longrightarrow 1..k : \forall j \in 1..k : \sum \{w_i : f(i) = j\} \leq b$$

BIN PACKING is NP-complete

Obviously BIN PACKING is in **NP**.

On the other hand PARTITION$\leq_p$BIN PACKING (Special instance):

$$(w_1,\ldots,w_k) \mapsto \begin{cases} \text{dozen size:} & b = \frac{1}{2}\sum_{i=1}^{k} w_i \\ \text{number:} & k = 2 \\ \text{weights:} & w_1,\ldots,w_k \end{cases}$$